1.

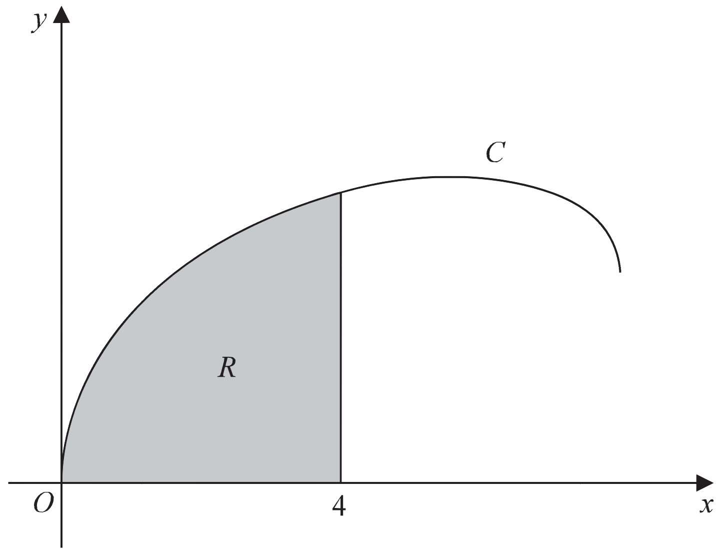

Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right) dt$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

(a) $R = \int_q^p y \frac{dx}{dt}\, dt$

$x = 8\sin^2 t$

$\frac{dx}{dt} = 8 \times 2\sin t\cos t$

$= 16\sin t \cos t$

$\frac{d}{dx}\sin^2 x = 2\sin x\cos x$ using the chain rule with $u = \sin x$

$y \times \frac{dx}{dt} = (2\sin 2t + 3\sin t) \times 16\sin t\cos t$ 1

$$y \times \frac{dx}{dt} = \left[2(\sin t\cos t + \cos t \sin t) + 3\sin t\right] \times 16 \sin t \cos t$$

$$= (4\sin t \cos t + 3\sin t) \times 16 \sin t \cos t$$

$$= (64\sin^2 t \cos t^2 + 48 \sin^2 t \cos t)$$

$$R = \int_0^a y\frac{dx}{dt}\,dt = \int_0^a 64\sin^2 t\cos t^2 + 48\sin^2 t\cos t \; dt \quad 1$$

We need to find a way to simplify this into the required form.

$$\begin{aligned}
\cos 4t &= 2\cos^2 2t - 1 \\
&= 2(1-\sin^2 2t) - 1 \\
&= 2 - 2\sin^2 2t - 1 \\
&= 1 - 2\sin^2 2t \\
&= 1 - 2(\sin 2t \sin 2t) \\
&= 1 - 2(2\sin t\cos t \times 2\sin t \cos t) \\
&= 1 - 2(4\sin^2 t\cos^2 t) \\
&= 1 - 8\sin^2 t\cos^2 t \quad 1
\end{aligned}$$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

using $\sin(2x) = 2\sin x\cos x$

$$8\sin^2 t\cos^2 t = 1 - \cos 4t$$

$$64\sin^2 t\cos^2 t = 8(1-\cos 4t)$$

$$R = \int_0^a 8 - 8\cos 4t + 48\sin^2 t\cos t \; dt \quad 1$$

$$a = \frac{\pi}{4} \quad 1$$

Finding the new domain :

$$R = \int_0^4 y\,dx = \int y\frac{dx}{dt}\,.\,dt$$

$$x = 8\sin^2 t \longrightarrow \frac{dx}{dt} = 16\sin t\cos t$$

when $x = 0$, $8\sin^2 t = 0$, Hence, $t = 0$

when $x = 4$, $8\sin^2 t = 4 \longrightarrow \sin^2 t = \frac{1}{2}$

$$t = \sin^{-1}\sqrt{\tfrac{1}{2}} = \pi/4$$

(b) $\int_0^{\frac{\pi}{4}} 8 - 8\cos 4t + 48\sin^2 t \cos t \; dt = 8t - 2\sin 4t + 16\sin^3 t$ (2)

$\left[8t - 2\sin 4t + 16\sin^3 t \right]_0^{\frac{\pi}{4}} = \left[8\left(\frac{\pi}{4}\right) - 2\sin\left(4 \times \frac{\pi}{4}\right) + 16\sin^3\left(\frac{\pi}{4}\right) \right]$

$- \left[8(0) - 2\sin(4 \times 0) + 16\sin^3(0) \right]$ (1)

$= 2\pi + 4\sqrt{2}$ (1)

2. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

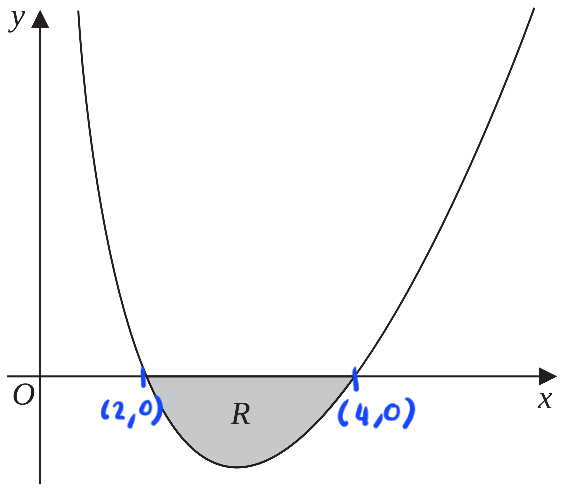

Figure 3

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \qquad x > 0$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

Find the exact area of R, writing your answer in the form $a\sqrt{2} + b$, where a and b are constants to be found.

(6)

$y = 0 : \dfrac{(x-2)(x-4)}{4\sqrt{x}} = 0$

$(x-2)(x-4) = 0$

$x = 2, 4$ (1)

$y = \dfrac{(x-2)(x-4)}{4\sqrt{x}}$ (1)

$y = \dfrac{x^2 - 6x + 8}{4x^{\frac{1}{2}}}$

$y = \dfrac{x^2}{4x^{\frac{1}{2}}} - \dfrac{6x}{4x^{1/2}} + \dfrac{8}{4x^{1/2}}$ (1)

$$y = \frac{x^{3/2}}{4} - \frac{3x^{1/2}}{2} + 2x^{-1/2}$$

$$\therefore \int_2^4 \frac{(x-2)(x-4)}{4\sqrt{x}}\,dx = \int_2^4 \left\{ \frac{x^{3/2}}{4} - \frac{3x^{1/2}}{2} + 2x^{-1/2} \right\} dx \quad (1)$$

$$= \left[\frac{x^{5/2}}{10} - x^{3/2} + 4x^{1/2} \right]_2^4 \quad (1)$$

$$= \left\{ \frac{4^{5/2}}{10} - 4^{3/2} + 4 \times 4^{1/2} \right\} - \left\{ \frac{2^{5/2}}{10} - 2^{3/2} + 4 \times 2^{1/2} \right\}$$

$$= \left(\frac{32}{10} - 8 + 8 \right) - \left(\frac{2}{5}\sqrt{2} - 2\sqrt{2} + 4\sqrt{2} \right)$$

$$= \frac{16}{5} - \frac{12}{5}\sqrt{2}$$

(1)

$$\text{Area } R = \frac{16}{5} - \frac{12}{5}\sqrt{2}$$